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Extension of Relativistic Kinematics

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We give an interpretation of the standard coordinates on Minkowski space time that permits us to introduce an additional invariant time concept. We then consider an alternative set of coordinates and kinematical conservation laws on the resulting "space-time." We obtain two conservation laws in addition to those of relativistic kinematics. The new conservation laws concern the quantities kinematical mass and mass defect, which we have introduced in order to have a more complete description of composite systems.

1. INTRODUCTION

The construction of a particle dynamics is ultimately based on assumptions about structure of space-time and kinematics as perceived by a class of acceptable observers. Some of these assumptions are quite general and do not impose any detailed structure, while others discriminate between different *a priori* possibilities like Galilei or Einstein relativity.

In the present paper we will discuss these concepts within the framework set by the theory of special relativity. We start by considering the (Minkowski) space-time M and the interpretation of the canonical coordinates. This discussion permits us to define a universal "invariant" time which might serve as the parameter measuring the *evolution* of, and the *correlation* between, the states of the particles constituting a composite system.

The introduction of the additional time lead us to introduce the extended configuration space $M \times \mathbb{R}$, on which we define another set of coordinates. These coordinates are instrumental for the subsequent discussion of the kinematics. In fact, they suggest our choice of momentum space of a particle along with the action of the kinematical symmetry group.

Our definition of a free relativistic particle is given such as to be general enough to include the center of mass system of a system of several

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particles. To this end we have introduced the concept of kinematical mass which is the sum of the masses of the constituents and differs from the effective mass or invariant energy (c.m. energy) to which it is related by a constraint to the "mass shell." We also introduce a concept of "internal energy" as a measure for the effective mass.

In our choice of definitions we have been conducted by the following principles:

(i) One free particle should be described in the same way as the center of mass of a system of several particles.

(ii) The usual Einstein relativistic kinematical conservation laws should be incorporated.

(iii) The formulation should be compatible with the claim that the relative localization of two particles is a space-time localization.

Considered *a posteriori*, these conditions seems to limit quite effectively the kind of theory one might have.

In this paper, we have limited ourselves to considerations about the kinematical conservation laws, and the interpretation of the Minkowski space-time coordinates and universal time. Discussions of the nontrivial dynamics, classical and quantal, can be found in Aaberge (1977a, 1977b, 1983).

2. MINKOWSKI SPACE AND THE RESTRICTED INHOMOGENEOUS LORENTZ GROUP

The mathematical structure of the theory of special relativity is summarized in the isometric action of the restricted inhomogeneous Lorentz group on Minkowski space.

Minkowski space (M, g) is a four-dimensional manifold M homeomorphic to $\mathbb{R}^4$ and endowed with a semi-Riemannian metric g which in the canonical orthonormal coordinates $z^\mu: M \to \mathbb{R}^4$ is represented by the matrix

$$(g_{\mu\nu}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The restricted inhomogeneous Lorentz group is the semidirect product of the semisimple connected group $SO(3.1)_0$[1] and the Abelian group $\mathbb{R}^4$. In its action on M it appears as the connected component of the group of isometries of (M, g).

[1]$SO(3.1)_0 \equiv$ connected component of $SO(3.1)$.

The interpretation of the restricted inhomogeneous Lorentz group $SO(3.1)_0 \times_s \mathbb{R}^4$ as a relativity group leads to the parametrization

$$\{(\theta^i)\} = SO(3) \subset \{(\theta^i, u^i)\} = SO(3,1)_0 \subset \{(\theta^i, u^i, a^\mu)\} = SO(3,1)_0 \times_s \mathbb{R}^4$$

We will denote by

$$SO(3,1)_0 \times_s \mathbb{R} \to \mathrm{Aff}(\mathbb{R}^4)$$

$$(\theta^i, u^i, a^\mu) \mapsto \left(\Lambda(\theta^i, u^i)^\mu{}_\nu, a^\mu\right)$$

the corresponding affine representation of $SO(3,1)_0 \times_s \mathbb{R}^4$ on $M \equiv \mathbb{R}^4$, and by

$$z^\mu \mapsto \Lambda(\theta^i, u^i)^\mu{}_\nu z^\nu + a^\mu$$

the associated action on the canonical *orthonormal coordinates*. The matrix (Λ^μ_ν) then has the form

$$\Lambda(0^i, u^i)^0{}_0 = \gamma$$

$$\Lambda(0^i, u^i)^i{}_0 = \Lambda(0^i, u^i)^i{}_0 = \gamma u^i/c$$

$$\Lambda(0^i, u^i)^i{}_j = \delta^i{}_j + \frac{\gamma^2}{\gamma+1}\frac{u^i u_j}{c^2}$$

with $\gamma = (1-(u^i)^2/c^2)^{-1/2}$, where c is the velocity of light *in vacuo*;

$$\Lambda(\theta^i, 0^i)^0{}_0 = \Lambda(\theta^i, 0^i)^0{}_i = \Lambda(\theta^i, 0^i)^i{}_0 = 0$$

and $(\Lambda(\theta^i, 0^i)^i_j)$ is the usual representation of the rotation group $SO(3)$ on $\mathbb{R}^3$.

The structure of $SO(3.1)_0$ is moreover, completely determined by the two relations

$$\Lambda(\phi^i, 0^i)\Lambda(\theta^i, u^i)\Lambda^{-1}(\phi^i, 0^i) = \Lambda\Big(\Lambda(\phi^i, 0^i)^i{}_j\theta^j, \Lambda(\phi^i, 0^i)^i{}_j u^j\Big)$$

and

$$\mathsf{L}^{-1}\big(w^\mu(v^i)\big)\Lambda(0^i, u^i)\mathsf{L}\big(\Lambda(0^i, u^i)^\mu{}_\nu w^\nu(v^i)\big)$$

$$= \Lambda\Big(\Phi^i_w\big(w^\mu((v^i), \Lambda^{-1}(0^i, u^i)\big), 0^i\Big)$$

where

$$(w^{\mu}(v^i)) = \left(\left(1-(v^i)^2/c^2\right)^{-1/2}c, \left(1-(v^i)^2/c^2\right)^{-1/2}v^i\right)$$

and

$$\mathsf{L}(w^{\mu}(v^i)) = \Lambda(0^i, v^i)$$

3. THE EXTENDED CONFIGURATION SPACE

To a given inertial observer λ, physical space and time have separate realities. A distance in physical space is measured with a "measuring stick," and a duration in time with a clock. Every event recorded by the observer λ can thus be classified according to its relative position in physical space (a^i), and the time of its occurrence t, by means of λ's "measuring stick" and clock C.

Assume that λ observes a clock C' moving with constant velocity (u^i) relative to himself, at the position (a^i) at time $t = 0$, and that moreover, he reads off the time a^0/c on C'. Then, at time t, λ would find the clock C' at

$$z^i(t) = \gamma u^i t + a^i$$

and he would read off the "time"

$$z^0(t)/c = \gamma t + a^0/c$$

According to the theory of special relativity, the "motion" of the clock C' is described by a line, its worldline, in the Minkowski space M. With respect to a canonical orthonormal coordinatization "relative to λ" this line is given by

$$t \mapsto (z^{\mu}(t)) = (w^{\mu}(u^i)t + a^{\mu})$$

One can also represent the motion of C' by the curve

$$t \mapsto (z^{\mu}(t), t)$$

in extended configuration space $M \times \mathbb{R}$, on which we *assume* the action

$$(z^{\mu}, t) \mapsto (\Lambda(\theta^i, u^i)^{\mu}{}_{\nu} z^{\nu} + a^{\mu}, t) \tag{1}$$

This description of the motion of a clock C is manifestly covariant. Moreover, observing that each side of the identity

$$t^2c^2 = \left(z^\mu(t) - a^\mu\right)\left(z_\mu(t) - a_\mu\right)$$

is Lorentz invariant, we are naturally led to the following interpretation of the action (1) and time t. Consider two inertial observers λ and λ' who are equipped with the clocks C and C'. To λ, the motion of C is described by

$$t \mapsto (ct, 0, 0, 0, t)$$

and the motion of C' by

$$t \mapsto \left(\gamma ct + a^0, \gamma u^i t + a^i, t\right)$$

The description relative to λ' is obtained via the Lorentz transformation $(\Lambda(0^i, -u^i), -\Lambda(0^i, u^i)^\mu_\nu a^\nu)$ which relates λ and λ'; thus, with respect to λ', the motion of C is described by

$$t \mapsto \left(\gamma ct - \gamma\left(a^0 - a^j u^j/c\right), -\gamma u^i t - \left(a^i + \frac{\gamma}{\gamma+1}\frac{a^j u^j}{c^2}u^i - \frac{\gamma a^0 u^i}{c}\right), t\right)$$

and the motion of C' by

$$t \mapsto (ct, 0, 0, 0, t)$$

Accordingly, in the description relative to λ, t appears as the time measured by the clock C as observed by λ, and in the description relative to λ' it is the time measured by the clock C' as observed by λ'. In other words,

> t must be interpreted as the time measured by the clock of the observer relative to which the system considered is described, as observed by this observer.

An alternative set of coordinates is defined on extended configuration space by

$$(z^\mu, t) \mapsto (q^\mu, t) = (z^0 - ct, z^i, t)$$

In these coordinates which are coordinates relative to the frame of reference, the action (1) of $SO(3,1)_0 \underset{s}{\times} \mathbb{R}^4$ reads

$$(q^\mu, t) \mapsto \left(\Lambda(\theta^i, u^i)^\mu{}_\nu q^\nu + v^\mu(u^i)t + a^\mu, t\right)$$

for

$$v^\mu(u^i) = ((\gamma - 1)c, u^i)$$

Accordingly, the motion of the clock C' is described by λ in the coordinates (q^μ, t) by

$$t \mapsto (q^\mu(t), t) = (v^\mu(u^i)t + a^\mu, t)$$

The expression $v^\mu(u^i)$ will be called the four-velocity of the clock C' relative to λ. It turns out that with this definition of four-velocity, the kinetic energy of the clock C' relative to λ, can be given in the following suggestive form:

$$T = (\gamma - 1)mc^2 = \tfrac{1}{2}mv^\mu(u^i)v_\mu(u^i)$$

where m denotes the restmass assigned to C'.

In the following we will consider the kinematics of Einstein relativistic particles in these coordinates.

4. THE ONE-PARTICLE SYSTEM

Definition. The system of a free Einstein relativistic particle of kinematical mass m is by assumption associated with

(i) the momentum space

$$M = \{ p^\mu \in \mathbb{R}^4 | (p^0 + mc)^2 - (p^i)^2 > 0 \;\&\; p^0 > -mc \}$$

(ii) the kinematical symmetry group $SO(3,1)_0 \underset{s}{\times} \mathbb{R}^4$ acting on M by

$$(\Lambda(\theta^i, u^i)^\mu{}_\nu, a^\mu) : p^\mu \mapsto \Lambda(\theta^i, u^i)^\mu{}_\nu p^\nu + mv^\mu(u^i)$$

(iii) the observables four-momentum $\boldsymbol{p}^\mu$, mass defect $\boldsymbol{\Delta m}$, effective mass $\boldsymbol{m}'$, internal energy $\boldsymbol{h}$, Hamiltonian $\boldsymbol{H}$, and "constraint" $\boldsymbol{\alpha}$, being represented by the following functions $M \to \mathbb{R}$:

$$\hat{p}^\mu(p^\mu) = p^\mu$$

$$\Delta\hat{m}(p^\mu) = \frac{1}{c}\left((p^0 + mc)^2 - (p^i)^2\right)^{1/2} - m$$

$$\hat{m}'(p^\mu) = \Delta\hat{m}(p^\mu) + m = \frac{1}{c}\left((p^0 + mc)^2 - (p^i)^2\right)^{1/2}$$

$$\hat{h}(p^\mu) = e \qquad (e > -\tfrac{1}{2}mc)$$

$$\hat{H}(p^\mu) = \frac{p^\mu p_\mu}{2m} + \hat{h}(p^\mu)$$

$$\hat{\alpha}(p^\mu) = p^0 - \frac{1}{c}\hat{H}(p^\mu)$$

$$= \Delta\hat{m}(p^\mu)c + \frac{\Delta\hat{m}(p^\mu)^2}{2m}c - \frac{1}{c}\hat{h}(p^\mu)$$

(iv) the constraint "to the mass shell"

$$\hat{\alpha}(p^\mu) = 0$$

A first consequence of this definition is that $\hat{H}$ and $\hat{p}^0 c$ has the same spectrum on the constrained momentum space (mass shell),

$$M^{(0)} = \{M \mid \alpha = 0\}$$

We will refer to this spectrum as the energy spectrum of the system. To prove this statement, consider the diffeomorphism

$$\Phi\colon M \to M, \qquad (p^0, p^i) \mapsto (\alpha, p^i)$$

It is easy to verify that

$$(\hat{p}^0 \circ \Phi)(\alpha, p^i) = \left((p^i)^2 + m^2\left(1 + \frac{2e}{mc^2}\right)c^2 + 2m\alpha c^2\right)^{1/2} - mc$$

$$(\hat{H} \circ \Phi)(\alpha, p^i) = c\left((p^i)^2 + m^2\left(1 + \frac{2e}{mc^2}\right)c^2 + 2m\alpha c^2\right)^{1/2} - mc^2 - \alpha c$$

thus,

$$(\hat{p}^0 \circ \Phi)(0, p^i)c = (\hat{H} \circ \Phi)(0, p^i)$$

$$= c\left((p^i)^2 + m^2\left(1 + \frac{2e}{mc^2}\right)c^2\right)^{1/2} - mc^2$$

For $e = 0$, this gives the usual expression for the kinetic energy of a free Einstein relativistic particle of mass m as a function of the momentum p^i.

For $e \neq 0$, $(p^0 \circ \Phi)(0, p^i)$ is naturally interpreted as the kinetic energy of a free Einstein relativistic particle of kinematical mass $m(1+2e/mc^2)^{1/2}$ the lower bound of whose energy spectrum has been shifted by the amount $((1+2e/mc^2)^{1/2}-1)mc^2$; or as the kinetic energy of free Einstein relativistic particle of kinematical mass m and internal energy e.

The validity of these two interpretations is based on the assumption that the lower bound of the energy spectrum is not uniquely defined, and they reflect the fact that in the Einstein relativistic case, a translation of the energy spectrum implies a redefinition of the kinematical mass, the mass defect and the internal energy, though leaving invariant the effective mass. Such a translation, by an amount Δe is consistently defined on $\{(m, p^\mu, \Delta m, h, H)\}$ by the semigroup action

$$\begin{aligned}
m &\mapsto m + \Delta e/c^2 \\
p^0 &\mapsto p^0 - \Delta e/c \\
p^i &\mapsto p^i \\
\Delta m &\mapsto \Delta m - \Delta e/c^2 \\
h &\mapsto \frac{m}{m+\Delta e/c^2} h + \frac{1}{2}\left(\frac{m^2}{m+\Delta e/c^2} - m - \frac{\Delta e}{c^2}\right)c^2 \\
H &\mapsto \frac{m}{m+\Delta e/c^2}\left(H - \frac{p^0 \Delta e}{c} - \frac{\frac{1}{2}\Delta e^2}{c^2}\right) - \frac{1}{2}\left[\frac{m^2}{m+\Delta e/c^2} - \left(m + \frac{\Delta e}{c^2}\right)\right]c^2
\end{aligned} \tag{2}$$

Accordingly, the effective mass $m' = \Delta m + m$ appears as an intrinsic characteristic of the particle, while the definition of kinematical mass and internal energy depend on the choice of lower bound of the energy spectrum.

5. THE SYSTEM OF TWO PARTICLES

Definition. The system of two free Einstein relativistic particles of kinematical masses m_1 and m_2 ($m_1 \geqslant m_2$) is by assumption associated with

(i) the momentum space

$$M \times M = \left\{(p_1^\mu, p_2^\mu) \in \mathbb{R}^8 \mid (p_n^0 + m_n c)^2 - (p_n^i)^2 > 0 \right.$$
$$\left. \text{and } p_n^0 > -m_n c, n = 1,2\right\}$$

(ii) the kinematical symmetry group $SO(3.1)_0 \underset{s}{\times} \mathbb{R}^4$ acting on $M \times M$ by

$$p_n^\mu \mapsto \Lambda(\theta^i, u^i)^\mu{}_\nu p_n^\nu + m_n v^\mu(u^i) \qquad (n=1,2)$$

(iii) the individual particle observables $\boldsymbol{p}_1^\mu, \boldsymbol{p}_2^\mu, \Delta m_1, \Delta m_2$ etc. which are supposed realized as in section 4; and, the observables of momentum $\boldsymbol{P}^\mu$ and mass defect $\boldsymbol{\Delta M}$ of the center of mass, momentum $\boldsymbol{p}^\mu$ of the internal system, the Hamiltonian $\boldsymbol{H}$, the internal Hamiltonian $\boldsymbol{h}$, and the "constraints" $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which are defined by the following functions $M \times M \to \mathbb{R}$:

$$\hat{P}^\mu \circ \Psi(p_1^\mu, p_2^\mu) = p_1^\mu + p_2^\mu$$

$$\Delta\hat{M} \circ \Psi(p_1^\mu, p_2^\mu) = \frac{1}{c}\left(\left(p_1^0 + p_2^0 + Mc\right)^2 - \left(p_1^i + p_2^i\right)^2\right)^{1/2} - M$$

$$\hat{p}^\mu \circ \Psi(p_1^\mu, p_2^\mu) = L^{-1}(p_1^\mu + p_2^\mu)^\mu{}_\nu \frac{m_1 p_2^\nu - m_2 p_1^\nu}{m_1 + m_2}$$

$$\hat{H} \circ \Psi(p_1^\mu, p_2^\mu) = \frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} + e_1 + e_2$$

$$\hat{h} \circ \Psi(p_1^\mu, p_2^\mu) = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left(\frac{m_1 p_2^\mu - m_2 p_1^\mu}{m_1 + m_2}\right) \times \left(\frac{m_1 p_{2\mu} - m_2 p_{1\mu}}{m_1 + m_2}\right) + e_1 + e_2$$

$$\hat{\alpha} \circ \Psi(p_1^\mu, p_2^\mu) = p_1^0 + p_2^0 - \frac{1}{c}\hat{H} \circ \Psi(p_1^\mu, p_2^\mu)$$

$$\hat{\beta} \circ \Psi(p_1^\mu, p_2^\mu) = \hat{p}^0 \circ \Psi(p_1^\mu, p_2^\mu) - \frac{(1/c)(1-4m/M)^{1/2}\hat{h} \circ \Psi(p_1^\mu, p_2^\mu) - (m_1 e_1 - m_2 e_2)}{c\left(1+2\hat{h} \circ \Psi(p_1^\mu, p_2^\mu)/Mc^2\right)^{1/2}}$$

where

$$M = m_1 + m_2, \qquad m = \frac{m_1 m_2}{m_1 + m_2}, \qquad \text{and } L(P^\mu) = \Lambda\left(\frac{p^i c}{p^0 + Mc}\right)$$

(iv) the constraints "to the mass shells"

$$\hat{\alpha}_1(p_1^\mu, p_2^\mu) = 0 \qquad \text{and} \qquad \hat{\alpha}_2(p_1^\mu, p_2^\mu) = 0$$

or equivalently

$$\hat{\alpha}(p_1^\mu, p_2^\mu) = 0 \qquad \text{and} \qquad \hat{\beta}(p_1^\mu, p_2^\mu) = 0$$

To discuss some consequences of this definition, we will consider another representation defined by the diffeomorphism

$$\Psi: M \times M \to M \times M, \qquad (p_1^\mu, p_2^\mu) \mapsto (P^\mu, p^\mu) \tag{3}$$

In the coordinates (P^μ, p^μ), the momentum space $M \times M$ is characterized by

$$M \times M = \Bigg\{ (P^\mu, p^\mu) \in \mathbb{R}^8 \,|\, (P^0 + Mc)^2 - (P^i)^2 > 0,\ P^0 > -Mc,$$

$$\frac{m_1}{M}\big((P^0 + Mc)^2 - (P^i)^2\big)^{1/2} > p^0 > \frac{m_2}{M}\big((P^0 + Mc)^2 - (P^i)^2\big)^{1/2}$$

$$\text{and} \quad \frac{m_1\big((P^0 + Mc)^2 - (P^i)^2\big) - 2m_2 M\big(\hat{h}(p^\mu) - e_1 - e_2\big)}{2M\big((P^0 + Mc)^2 - (P^i)^2\big)^{1/2}} > p^0$$

$$> -\frac{m_2\big((P^0 + Mc)^2 - (P^i)^2\big) - 2m_1 M\big(\hat{h}(p^\mu) - e_1 - e_2\big)}{2M\big((P^0 + Mc)^2 - (P^i)^2\big)^{1/2}} \Bigg\}$$

The action of $SO(3,1)_0 \underset{s}{\times} \mathbb{R}^4$ is, moreover, given by

$$P^\mu \mapsto \Lambda(\theta^i, u^i)^\mu{}_\nu P^\nu + Mv^\mu(u^i)$$

$$p^\mu \mapsto \Lambda\big(\theta_w^i(P^\mu, \Lambda(\theta^i, u^i))\big)^\mu{}_\nu p^\nu$$

for $\Lambda(\theta_w^i(P^\mu, \Lambda(\theta^i, u^i)))$ being the rotation defined by

$$\Lambda\big(\theta_w^i(P^\mu, \Lambda(\theta^i, u^i))\big)^\mu{}_\nu = L^{-1}\big(\Lambda(\theta^i, u^i)^\mu{}_\nu P^\nu + Mv^\mu(u^i)\big)^\mu{}_\alpha \times \Lambda(\theta^i, u^i)^\alpha{}_\beta L(P^\mu)^\beta{}_\nu \tag{4}$$

The observables $\boldsymbol{p}_1^\mu$, $\boldsymbol{p}_2^\mu$, $\boldsymbol{P}^\mu$, ΔM, $\boldsymbol{p}$, $\boldsymbol{h}$, $\boldsymbol{H}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are represented by the functions

$$\hat{p}_1^\mu \circ \Psi^{-1}(P^\mu, p^\mu) = \frac{m_1}{M} P^\mu - L(P^\mu)^\mu{}_\nu p^\nu$$

$$\hat{p}_2^\mu \circ \Psi^{-1}(P^\mu, p^\mu) = \frac{m_2}{M} P^\mu + L(P^\mu)^\mu{}_\nu p^\nu$$

$$\hat{P}^\mu(P^\mu, p^\mu) = P^\mu$$

$$\Delta\hat{M}(P^\mu, p^\mu) = \frac{1}{c}\left((P^0 + Mc)^2 - (P^i)^2\right)^{1/2} - M$$

$$\hat{h}(P^\mu, p^\mu) = \frac{p^\mu p_\mu}{2m} + e_1 + e_2$$

$$\hat{H}(P^\mu, p^\mu) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + e_1 + e_2$$

$$\hat{\alpha}(P^\mu, p^\mu) = P^0 - \frac{1}{c}\hat{H}(P^\mu, p^\mu)$$

$$\hat{\beta}(P^\mu, p^\mu) = p^0 - \frac{(1/c)(1-4m/M)^{1/2}\hat{h}(p^\mu) - (1/Mc)(m_1 e_1 - m_2 e_2)}{\left[1 + 2\hat{h}(p^\mu)/Mc^2\right]^{1/2}}$$

The internal energy spectrum of the system is by definition the spectrum of $\hat{h}$ on the submanifold defined by the constraint $\hat{\beta} = 0$. Accordingly, the center of mass appears as a free Einstein relativistic particle of kinematical mass $M = m_1 + m_2$ and internal energy e, for every value e of the internal energy spectrum.

6. THE EINSTEIN RELATIVISTIC KINEMATICAL CONSERVATION LAWS

Let 1 and 2 denote two particles of kinematical masses m_1 and m_2, internal energies e_1 and e_2, and momenta p_1^μ and p_2^μ, entering into a "collision." In general a collision of two particles produces a number of particles $n = 3, \ldots, N$ of kinematical masses $m_n \neq 0$, internal energies e_n and momenta p_n^μ, and a number of "photons" $m = 1, \ldots, M$ of kinematical masses 0 and momenta k_m^μ.

Since the system is invariant under "spatial" and time translations we conclude that the following quantities are conserved during the collision:

$$p_1^\mu + p_2^\mu = \sum_{n=3}^{N} p_n^\mu + \sum_{m=1}^{M} k_m^\mu \tag{5}$$

or

$$p_1^0 + p_2^0 = \sum_{n=3}^{N} p_n^0 + \sum_{m=1}^{M} k_m^0 \tag{6}$$

$$p_1^i + p_1^i = \sum_{n=3}^{N} p_n^i + \sum_{m=1}^{M} k_m^i \tag{7}$$

and

$$\frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} = \sum_{n=3}^{N} \frac{p_n^\mu p_{n\mu}}{2m_n} + \sum_{m=1}^{M} ck_m^\mu + \Delta e \tag{8}$$

where $\Delta e = \sum_{n=3}^{N} e_n - e_1 - e_2$. Moreover, the assumption of the covariance of the conservation laws (5) implies the conservation of the kinematical mass

$$m_1 + m_2 = \sum_{n=3}^{N} m_n \tag{9}$$

In fact, the transformed of (5) by a Lorentz transformation is

$$\Lambda(\theta^i, u^i)^\mu{}_\nu p_1^\nu + m_1 v^\mu(u^i) + \Lambda(\theta^i, u^i)^\mu{}_\nu p_2^\nu + m_2 v^\mu(u^i)$$

$$= \sum_{n=1}^{N} \left(\Lambda(\theta^i, u^i)^\mu{}_\nu p_n^\nu + m_n v^\mu(u^i)\right) + \sum_{m=1}^{M} \Lambda(\theta^i, u^i)^\mu{}_\nu k_m^\nu$$

or because of (5)

$$(m_1 + m_2) v^\mu(u^i) = \sum_{n=3}^{N} m_n v^\mu(u^i)$$

which implies (9).

By adding the conservation laws (6) and (9) and changing to the coordinates $(\Delta m, p^i)$, one obtains

$$\left((p^i)^2+(\Delta m_1+m_1)^2c^2\right)^{1/2}+\left((p_2^i)^2+(\Delta m_2+m_2)^2c^2\right)^{1/2}$$
$$=\sum_{n=3}^{N}\left((p^i)^2+(\Delta m_n+m_n)^2c^2\right)^{1/2}+\sum_{m=1}^{M}\left((k_m^i)^2\right)^{1/2} \qquad (10)$$

Thus, the conservation laws (5) and (8) implies the Einstein relativistic conservation laws (7) and (10) (Hagedorn, 1963). In addition, we obtain the conservation law (9) and the law

$$\Delta m_1+\frac{\Delta m_1^2}{2m_1}+\Delta m_2+\frac{\Delta m_2^2}{2m_2}=\sum_{n=3}^{N}\left(\Delta m_n+\frac{\Delta m_n^2}{2m_n}\right)+\frac{\Delta e}{c^2}$$

which is obtained by subtracting (6) from (8) and change to the coordinates $(\Delta m, p)$. These additional conservation laws make it possible to obtain a more detailed description of nonelastic processes, i.e., imposes a certain mode of description which is compatible with the usual Einstein relativistic kinematics but not imposed by it.

Consider for example the description of the process in which an "atom" in a given initial stationary state of internal energy e_i decays to a final stationary state of internal energy e_f by emitting a "photon." In the center-of-mass frame of reference this process is described by the conservation laws

$$0=p_f^i+k_f^i$$

$$\Delta m_i+m=\left[(p_f^i)^2+(\Delta m_f+m)^2c^2\right]^{1/2}+\left[(k_f^i)^2\right]^{1/2}$$

$$\Delta m_i+\frac{\Delta m_i^2}{2m}=\frac{e_i}{c^2}$$

$$\Delta m_f+\frac{\Delta m_f^2}{2m}=\frac{e_f}{c^2}$$

From these expressions we obtain the following relation between the energy of the emitted "photon" in the center-of-mass frame of reference, and the internal energies of the initial and final states,

$$c\left[(k_f^i)^2\right]^{1/2}=\hbar\omega=\frac{e_i-e_f}{\left(1+2e_i/mc^2\right)^{1/2}}$$

A second example concerns the process of an annihilation of two particles. If we let m_1 denote the kinematical mass of the particle and e_2 the internal energy of the antiparticle, we can conclude from the conservation laws that the antiparticle has kinematical mass

$$m_2 = -m_1$$

and mass defect

$$\Delta m_2 = m_1\left[\left(1 - 2e_2/m_1c^2\right)^{1/2} + 1\right]$$

Moreover, assuming that the internal energy e_1 of the particle is equal to $-e_2$, we find that the particle and the antiparticle have the same effective mass,

$$\Delta m_1 + m_1 = \Delta m_2 + m_2$$

Another consequence is related to the description of particle production in collisions and decays. It is known from nuclear physics that there exists processes where the effective mass m' is not conserved. The mass conservation law (9) implies that the particles involved in such processes must be considered as composite systems whose kinematical mass is the sum m of the kinematical masses of its elementary constituents, and whose effective mass is $m' + \Delta m$, where Δm denotes the mass defect due to the binding energy. Such processes must thus be described in terms of a rearrangement of the elementary constituents, eventually together with pairs of constituent–anticonstituent.

7. THE ANTIPARTICLE

The above considerations justify the following:

Definition. The antiparticle of an Einstein relativistic particle of kinematical mass $m > 0$ and internal energy e is an Einstein relativistic particle of kinematical mass $-m$ and internal energy $-e$.

The formalism presented in Section 3 cannot be applied to describe the system of a particle and an antiparticle on the basis of this definition of the antiparticle, since $M = m_1 + m_2 = m - m = 0$. However, with respect to processes not involving the annihilation of the system, the kinematical mass and internal energy of the antiparticle can be redefined according to (2) with $\Delta e \neq 0$. In particular, the choice $\Delta e = 2mc^2$ makes the antiparticle appear as a particle of kinematical mass m and internal energy e.

REFERENCES

Aaberge, T. (1977a). *Helvetica Physica Acta*, **50**, 331.
Aaberge, T. (1977b). *Helvetica Physica Acta*, **50**, 917.
Aaberge, T. (1983). *International Journal of Theoretical Physics*, **22**, 723.
Hagedorn, R. (1963). *Relativistic Kinematics*. Benjamin, New York.